Mathcamp 2024 Qualifying Quiz Solutions

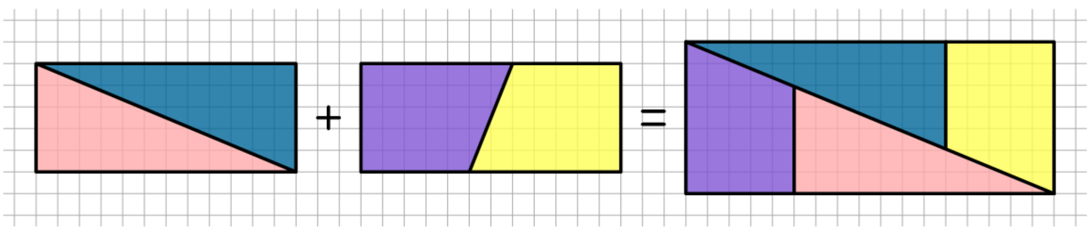
References

1a. Wikipedia contributors. "Chessboard paradox." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 4 Mar. 2024. Web. 6 Mar. 2024.

3. Bolander, Thomas, "Self-Reference", The Stanford Encyclopedia of Philosophy (Fall 2017 Edition)

Problem 1

The picture below shows a variant of a famous paradoxical puzzle. On the left, we take two rectangles of area 60, and cut each one into two pieces. On the right, we rearrange the four pieces, and put them together into a single rectangle of area 119. How could this be?



(a) Explain what's wrong. *(Similar paradoxes can be found under names such as "Chessboard paradox" or "Missing square puzzle". It's fine to look at a source that explains such a paradox, provided you cite it—and explain it in your own words.)*

Solution: We can start by examining the slopes of the line segments that seem to lay on the "diagonal" of the large rectangle. The slope of the line segments that are part of the trapezoids is $\frac{-2}{5}$ and the slope of the line segments that are part of the triangles is $\frac{-5}{12}$. This shows that the "diagonal" is in fact four distinct lines instead of what appears to be one, creating a parallelogram shaped space, with the line segments making up the "diagonal" as the sides of the parallelogram.

WLOG let the bottom left vertex be $(0, 0)$, the top left vertex be $(0, 7)$, the bottom right vertex be $(12, 0)$ and the top right vertex be $(12, 7)$. We can examine the halves on either side of the "diagonal" because of the symmetry present in the rectangle across the diagonal, so by finding the area of the one of the triangle shaped halves of the parallelogram, we can deduce the area of the parallelogram shaped empty space. The intersection point between the purple parallelogram's line segment and the pink triangle is $(5, 5)$. The vertices of the triangle have coordinates $(0, 7)$,$(5, 5)$,$(12, 0)$ and when applying the shoelace theorem, we can find that the area of the empty triangle is $\frac{1}{2}$, so the area of the empty parallelogram is 1. This empty area of area 1 is responsible for the paradox since without paradox the area would be $60 + 60 = 120$.

$\square$

(b) Although two 5×12 rectangles cannot *really* be rearranged into a 7×17 rectangle, it is possible to take two $a \times b$ rectangles and cut them as shown in the picture above to make a $c \times d$ rectangle, with no paradox. What should the lengths a, b, c, d be (up to scaling, of course)?

Solution: Let the shorter side of the rectangle be length a and the longer side of the rectangle be length b. Now the slope of the line segments of the trapezoids that lay on the diagonal have slope $-\frac{b-2a}{a}$ and the slope of the line segments of the triangles that lay on the hypotenuse is $-\frac{a}{b}$.

In order to have no paradox these slopes must be equal in order to have one line as the diagonal of the large rectangle, so we can set them equal and solve for one variable in terms of the other.

$$-\tfrac{b-2a}{a} = -\tfrac{a}{b}$$
$$b^2 - 2ab - a^2 = 0$$

Using the quadratic equation we can find that $a : b$ have a ratio of $a : a + a\sqrt{2}$ or $1 : 1 + \sqrt{2}$. Now we can solve for c and d in terms of a.

$$c = b - a = a + a\sqrt{2} - a = a\sqrt{2}$$
$$d = b + a = a + a + a\sqrt{2} = 2a + a\sqrt{2}$$

This means that the ratio of $a : b : c : d$ is

$$1 : 1 + \sqrt{2} : \sqrt{2} : 2 + \sqrt{2}$$

$\square$

(c) It is also possible to take *three* congruent rectangles, cut each one into two pieces, and rearrange them to form a single rectangle similar to the original three. How can we do this?

Try to find an answer that lets you create a paradoxical decomposition of your own!

Solution:

With this arrangement (Figure 1) we can have both a valid arrangement and a paradoxical arrangement. This is because by fixing d as a value, the paradox is dependent on the lengths of c and e since it determines whether the slopes of the green and blue trapezoids are the same. We can find the valid arrangement, with the valid length of

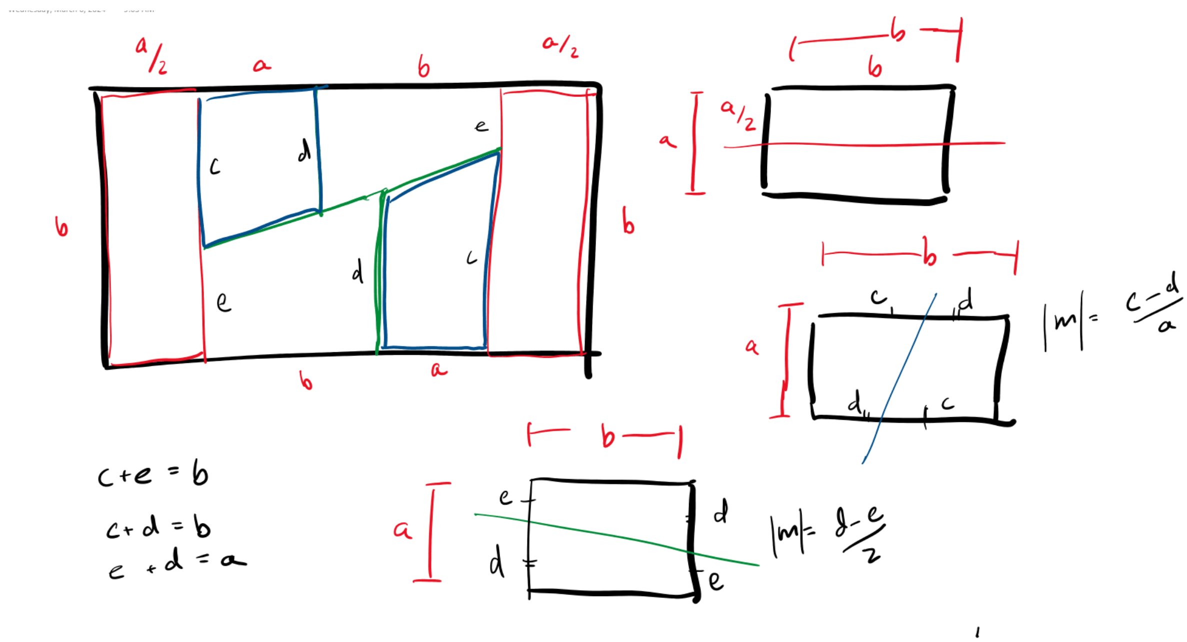


Figure 1: Arrangement of rectangles

the segments and then change the ratio of $c : e$ slightly to create the different but approximate slopes between the green and blue trapezoids, which will create the paradox.

We can first find the ratio of a and b by using the condition that the final rectangle and original rectangles must be similar. The length of the final rectangle is $4a$ and the width is b. We can set the ratio of the length to width of the final rectangle equal to the ratio of the length and width in the original rectangles.

$$\frac{2a+b}{b} = \frac{b}{a}$$

If we let $a = 1$ then we have:

$$(b-2)(b+1) = 0$$

So $b = 2$ since a length cannot be negative.

The value of a can be changed to anything since all the variables are dependent on a. We can change the value of a to any value since we now know that $a : b$ is $1 : 2$ so if we change a we can just do $b = 2a$ to find the new b.

We can now solve for the lengths of the line segments when the arrangement is nonparodoxical by setting the slopes of the trapezoid equal. We can use this equation as well as known lengths for the sums of the segments.

$c + e = 2$
$d + e = 1$
$d + c = 2$
$\frac{c-d}{1} = \frac{d-e}{2}$

Solving this system of equations yields us: $c = \frac{11}{6}, d = \frac{5}{6}, e = \frac{1}{6}$

We can just examine the half of the rectangle under the diagonal since both halves are symmetrical. The slopes of each of the trapezoids are $m_1 = \frac{\frac{5}{6}-e}{2}$ and $m_2 = \frac{c-\frac{5}{6}}{1}$. We can substitute e out of the first equation and we are left with:

$\frac{\frac{5}{6}-(2-c)}{2} \neq \frac{c-\frac{5}{6}}{1}$

We know must change c and e by a small increment to create an almost invisible difference between the two trapezoid slopes. We can let $e = \frac{35}{216}$ and $c = \frac{253}{216}$.

The slopes now are:

$m_1 = 0.3356481481$
$m_2 = 0.337962963$

$m_1 \neq m_2, m_1 \approx m_2$ which creates a paradox in the area because there is a slight concave present between the two segments of the "diagonal".

Observation: A fully general solution that will work for any a and b would involve having the width be $a\sqrt{3}$ and the length $b\sqrt{3}$ without b being dependent on a

$\square$

Problem 2

Kayla has two red boxes, two green boxes, and two blue boxes. Each of the six boxes contains a secret number. Kayla hands the boxes to Leo and asks him to write the letters A, a, B, b, C, and c on the boxes. We will write $\#A$, $\#a$, $\#B$, $\#b$, $\#C$, and $\#c$ to refer to the secret numbers inside these boxes.

From there, the boxes go to Maya, who opens the boxes and reports the differences $\#A-\#a$, $\#B-\#b$, and $\#C-\#c$ (in that order). Next, the boxes go to Nathan, who opens the boxes and reports the sum of the numbers in the red boxes, the sum of the numbers in the green boxes, and the sum of the numbers in the blue boxes (in that order). The resealed boxes, along with Maya's and Nathan's reports, are then handed back to Kayla, who must determine the numbers in each of the boxes.

Kayla expected Leo to label same-color boxes with the same letter, which would have made it easy for Kayla to figure the numbers: for example, knowing $\#A - \#a$ from Maya and $\#A + \#a$ from Nathan, Kayla could solve for $\#A$ and $\#a$. However, to Kayla's surprise, Leo is color blind, so his labeling had nothing to do with the colors.

(a) For which of the labelings that Leo used is it still possible for Kayla to determine the six secret numbers? (Kayla can see which colors have which labels on them.)

Solution: We can think of each of the numbers inside the boxes as variables that are assigned a color of box and a letter label. We can let these numbers be $x_1, x_2, x_3, ...x_6$.

The equations Maya gives, relate numbers of the same letter labelling to each other. The numbers Leo gives, relate numbers with the same color to each other. We can show this relationship in a graph, with the vertices being the numbers inside the boxes.

Let the variables $x_1, x_2, x_3, .., x_6$ be assigned where consecutive pairs of variables are of the same color. For example (x_1, x_2) are one color, (x_3, x_4) are another color, and (x_5, x_6) are another color. These connections between colors, made by Nathan's equations, will be green edges in the graph (Figure 2). The connection between letters, made by Maya's equations, will be red edges in the graph. The edges created by the connections between letters can be changed in order to create different arrangements.

In the graph, the cycles connecting the numbers, are an independent system of equations from the rest of the cycles in the graph. This is because variables present in one cycle will not be present in a different cycle since vertices are not shared between cycles. This means if all the cycles in the graph are solvable, the arrangement is solvable. We can create cases of various cycle sizes and determine if that case is valid, since if a cycle

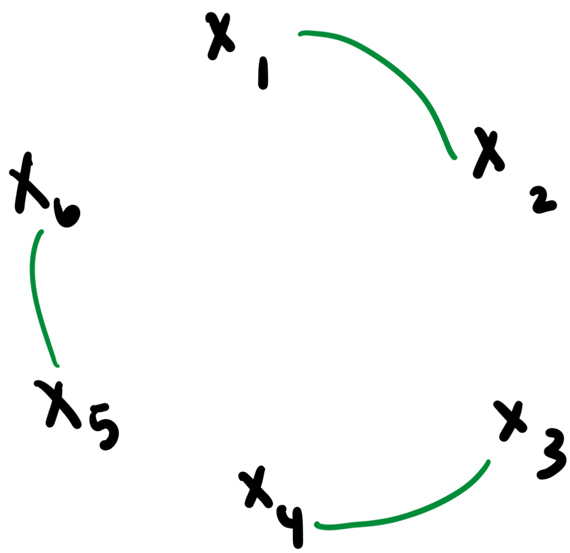


Figure 2: Nathan's connections in graph

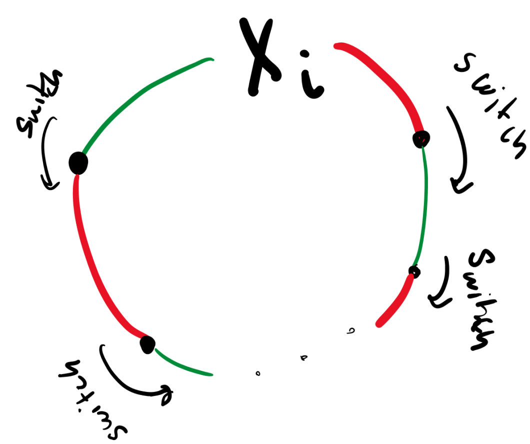


Figure 3: Graph showing there must be even amount of vertices in a cycle

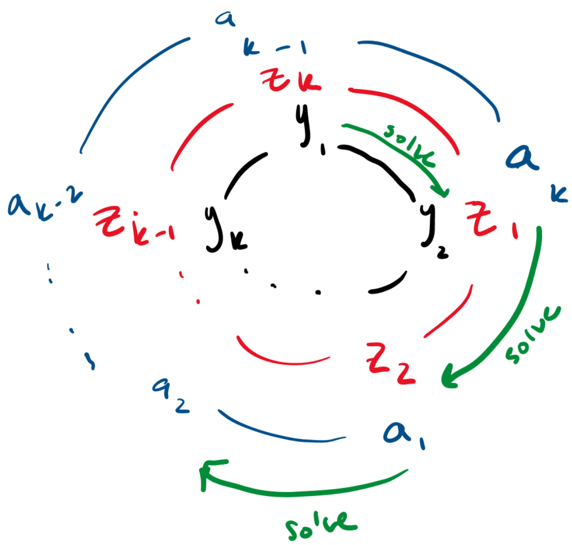


Figure 4: Variable relabelling to solve for next number

size of x is valid, all cycles of size x will be valid.

Lemma 1: All cycle sizes must be even.
Proof of Lemma 1: We can show this since each number must have only one of each color edges connecting it and the color of an edge alternates each vertices. Thus there must be an odd number of "switches" between the colors to get back to the same vertices which means there are an even number of vertices (Figure 3).

We can test the cases of even cycles: $2, 4, 6$
We can create a matrix with the system of equations for each of these cycle sizes. In order to see if the matrix is solvable we can try to eliminate the bottom most row.

Lemma 2: After the simplification of the matrix, if one of the variables is able to be solved, the rest of the vertices are able to be solved.
Proof of Lemma 2: We can re-label the k vertices in the cycle as $y_1, y_2, ..., y_k$ with y_i being connected by an edge to y_{i+1} except for y_k which is connected to y_1 instead of y_{k+1}. Assume y_1 is the variable we were able to solve for, since y_1 and y_2 share an equation, y_2 is able to be solved. We can then create a new labelling where the new z_1 is the old y_2 and new $z_i = y_{i+1}$ except for new z_k which is old y_1 (Figure 4). We can solve for z_1 and by result now solve for z_2. This solving and relabelling of the variables can be done $k - 1$ times to solve for the entire cycle.

Thus by solving for one variable, we know the entire cycle can be solved.

Size 2:

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Which can be simplified by doing the row operation of -R2+R1

$$\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$$

This system of equations is solvable.
Size 4:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Which can be simplified by doing -R2+R1 then -R3+R2 then R4+R3

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This system of equations cannot be solved, shown by the row of zeros in R4
Size 6:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Which can be simplified by doing -R2+R1 then -R3+R2 then R4+R3 then R5+R4 then -R6+R5

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This system of equations is solvable, so the valid cycle sizes are 2 and 6.

Thus arrangements that only have cycles of size 2 or cycles of size 6 will be valid. These arrangements are where either all pairs of letters are on same color boxes or no pairs of letters are on same color boxes.

□

(b) Generalize your answer to a problem with $2n$ boxes of n different colors, labeled with n different uppercase and lowercase letters.

Solution: We can think of each of the numbers inside the boxes as variables that are assigned a color and a number. We can let these numbers be $x_1, x_2, x_3, ...x_{2n}$.

We can create a graph in the same way we did in part A, that relate the numbers in Nathan and Maya's equations.

Claim: Only arrangements that have cycles of size S where $S \not\equiv 0 \pmod 4$ are solvable

Proof: Let $k = \lceil(\frac{S}{4})\rceil$ We can split the S equations from the graph into k groups by grouping, when possible, 4 equations together, in the same order they are shown in the graph starting with x_1. If S is not a multiple of 4, then the last group will be size $S - 4(\lceil(\frac{S}{4})\rceil - 1)$. Let the "last group", be the final group of equations remaining when the entire system is split into groups of 4 besides the group that connects x_S and x_1 which does not have to be of size 4. We can observe the properties of these groups.

From part A, we can see that we negated the second and third equations in a group of 4 equations, which caused all the variables to cancel out when added. We can apply this same pattern to each group, negating the second and third equations in a group. In the last group, we can negate the second and third equations if they are present.

i^{th} group

non-last group

$$\left.\begin{aligned} & x_{4(i-1)+1} + \cancel{x_{4(i-1)+2}} \\ & -(\cancel{x_{4(i-1)+2}} - \cancel{x_{4(i-1)+3}}) \\ & \quad -(\cancel{x_{4(i-1)+3}} + \cancel{x_{4(i-1)+4}}) \\ & \qquad (\cancel{x_{4(i-1)+4}} - x_{4(i-1)+5}) \end{aligned}\right\} = x_{4(i-1)+1} - x_{4(i-1)+5}$$

Figure 5: Reduction of ith non-last group

$$\left.\begin{aligned} & x_1 - \cancel{x_5} \\ & \cancel{x_5} - \cancel{x_9} \\ & \quad \vdots \\ & \quad \cancel{x_{4(k-3)+1}} - \cancel{x_{4(k-3)+5}} \\ & \qquad \cancel{x_{4(k-2)+1}} - x_{4(k-2)+5} \end{aligned}\right\} = x_1 - x_{4(k-2)+5}$$

Figure 6: 1 to k-1 th group reduction

First we observe what occurs to a size 4 group when all the equations are summed together. The system is reduced to the two term equation $x_{4(i-1)+1} - x_{4(i-1)+5}$, for the ith group (Figure 6). This pattern will continue for all of the non-last groups. Excluding x_1 and $x_{4(i-1)+5}$ where $i = k - 1$, all first terms in the reduced equations are equal to the negative of the second term in the previous equation since $x_{4(i-1)+5} = x_{4(i+1-1)+1} = x_{4i+1}$ Thus, all the reduced equations of each non-last group when summed, will be $x_1 - x_{4(k-2)+5}$ (Figure 5).

When $S \equiv 0 \pmod 4$, the last group will be of size 4 as well and the reduced equation of the group will be $x_{4(k-1)+1} - x_1$ (Figure 6). We can see that the non-last group simplified equation and the last group simplified equation will sum to 0, thus the cycle does not have one solution, and either has none or infinite solutions.

$$\begin{aligned} & x_{4(k-1)+1} + \cancel{x_{4(k-1)+2}} \\ & \quad -\cancel{x_{4(k-1)+2}} + \cancel{x_{4(k-1)+3}} \\ & \qquad -\cancel{x_{4(k-1)+3}} - \cancel{x_{4(k-1)+4}} \\ & \qquad\quad \cancel{x_{4(k-1)+4}} - x_1 \end{aligned}$$

Figure 7: $S \equiv 0 \pmod 4$ reduced kth group

$$\left.\begin{array}{l} x_{4(k-1)+1} + \cancel{x_{4(k-1)+2}} \\ \qquad - \cancel{x_{4(x-1)+2}} + x_1 \end{array}\right\} x_{4(k-1)+1} + x_1$$

Figure 8: $S \equiv 2 \pmod 4$ reduced kth group

When $S \equiv 2 \pmod 4$ the reduced equation of the last group will be, $x_{4(k-1)+1} + x_1$ (Figure 8). When the non-last group simplified equation and this last group equation are added, there is one valid solution, which means the cycle can be solved.

Since cycle size must be even, we have gone through all cases of remainders since even numbers are either $S \equiv 0 \pmod 4$ or $S \equiv 2 \pmod 4$

$\square$

Problem 3

Here is a curious fact you may not have known about mathematicians: when asked a yes-or-no question, number theorists will always tell the truth, while analysts will always lie. Be careful: if you ask someone a yes-or-no question, and they cannot answer either "yes" or "no" to it, then the universe explodes in paradox. For example, this happens if you ask an number theorist, "Is your answer to this question 'no'?"

(a) What yes-or-no question can be asked to both a number theorist and an analyst to cause the universe to explode in both cases?

Solution: Claim: "Will a number theorist respond "no" to this question?"

Proof: We can first examine what happens to the number theorist when they are asked this question.

Since the number theorist can only answer either "yes" or "no", if we show both of these responses are lies and create a self-referential paradox, than the universe will explode in paradox. If the number theorist where to answer "yes" to the question, it would not be the truth since the number theorist is not responding "no". If the number theorist were to answer "no", this would also be a lie since the number theorist is actually responding "no". Since both responses are lies, this question will create paradox and cause the universe to explode.

When asking the question to an analyst, the question refers to the response from the number theorist to the same question. We already proved that the number theorist will not be able to have a non-paradoxical response, so analyst will not be able to answer either since the question is asking about the response from number theorist. Since the analyst is not able to answer, this will cause the universe to explode in paradox.

□

(b) To try to prevent the universe from exploding in paradox, logicians have decided to act as paradox detectors. When you ask a logician a yes-or-no question, the logician will answer "yes" if asking the question to a number theorist would cause the universe to explode, and "no" otherwise. For example, if you ask a logician, "Is your answer to this question 'no'?" the logician will answer "yes".

What yes-or-no question can be asked to a logician to cause the universe to explode?

Solution: Claim: "Will logicians say yes to "will number theorists say no to this question?"?"

Proof: If we ask this question to a number theorist, since they always tell the truth, they would answer "yes". This is because the question "will number theorists say no to this question?" creates a self-referential paradox for the number theorist which is the criteria for a logician to say "yes". This would be paradox, since if the number theorist said, "yes" to the question, it did not make the universe explode so the logician would actually answer "no". If the logician answered, "no" this would not be true since the number theorist always tells the truth and it said that the logician will always say "yes". Thus this question will create a paradox, and causes the universe to explode.

□

Problem 4

A group of 11 Mathcampers decided to bake a rainbow unicorn sprinkle cake with 10 slices. Unfortunately they cannot split up the slices into smaller pieces, because they do not want to risk upsetting the unicorn. Each Mathcamper wants a slice of the cake (but not more, because Mathcampers aren't greedy). Each Mathcamper helped bake the cake to some extent. The Mathcampers have been assigned fractions $x_1, x_2, \ldots, x_{11}$ representing their share of the credit in baking the cake, where $0 < x_i < \frac{1}{10}$ for each i, and $x_1 + x_2 + \cdots + x_{11} = 1$.

(a) How can you randomly distribute the slices of cake such that the i^{th} Mathcamper has a probability of exactly $10x_i$ of getting a slice of cake?

Solution: We can approach this problem by using the probabilities of each combination of distributions of slices to the students.

Let $C(i, ...)$ be the combination where students $i, ...$ receive a piece of cake. Let $P(i, ...)$ be the probability of this combination occurring. Looking at $n = 3$, $k = 2$ we can see that $x_i = P(i, ...) + P(i...)$ since the probability of a person getting cake is equal to the sum of the probabilities of the combinations where they get a slice of cake, which would be 2 combinations for each student. We can create a system of equations for this case:

$$2x_1 = P(1,2) + P(1,3)$$
$$2x_2 = P(1,2) + P(2,3)$$
$$2x_3 = P(1,3) + P(2,3)$$

Solving for $P(1,2)$ we get $x_1 + x_2 - x_3 = P(1,2)$
Solving for $P(1,3)$ we get $x_1 + x_3 - x_2 = P(1,3)$
Solving for $P(2,3)$ we get $x_2 + x_3 - x_1 = P(2,3)$

We can see a pattern that all the probabilities of the kids that are receiving cake are summed, while the kids that are not receiving cake have their probability subtracted.

When creating this same system of equations for $n = 11, k = 10$:

$$P(1,2,3,4,5,6,7,8,9,10) = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} - x_{11}$$
$$P(1,2,3,4,5,6,7,8,9,11) = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{11} - x_{10}$$
$$P(1,2,3,4,5,6,7,8,10,11) = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_{10} + x_{11} - x_9$$
$$P(1,2,3,4,5,6,7,9,10,11) = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_9 + x_{10} + x_{11} - x_8$$

$$
\begin{aligned}
P(1,2,3,4,5,6,8,9,10,11) &= x_1+x_2+x_3+x_4+x_5+x_6+x_8+x_9+x_{10}+x_{11}-x_7\\
P(1,2,3,4,5,7,8,9,10,11) &= x_1+x_2+x_3+x_4+x_5+x_7+x_8+x_9+x_{10}+x_{11}-x_6\\
P(1,2,3,4,6,7,8,9,10,11) &= x_1+x_2+x_3+x_4+x_6+x_7+x_8+x_9+x_{10}+x_{11}-x_5\\
P(1,2,3,5,6,7,8,9,10,11) &= x_1+x_2+x_3+x_5+x_6+x_7+x_8+x_9+x_{10}+x_{11}-x_4\\
P(1,2,4,5,6,7,8,9,10,11) &= x_1+x_2+x_4+x_5+x_6+x_7+x_8+x_9+x_{10}+x_{11}-x_3\\
P(1,3,4,5,6,7,8,9,10,11) &= x_1+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10}+x_{11}-x_2\\
P(2,3,4,5,6,7,8,9,10,11) &= x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10}+x_{11}-x_1
\end{aligned}
$$

This pattern is present in the $n = 11, k = 10$ case as well which leads us to the following claim.

Claim: We can assign probabilities to each of the possible combinations of ways to give 10 slices of cake to 11 kids. We can then create sectors on a roulette wheel for each of the 11 combinations, with each sectors area proportional to each combinations probability. We can then spin this roulette wheel to determine which combination of kids will receive a slice of cake. Each of these probabilities for each combination $(i_1, ...i_k)$ of kids that receive cake is $(\sum_{j=1}^{n} x_j) - x_m, j \in i_1, ...i_k, m \notin i_1, ..., i_k$

Proof: In order to prove that this method is a valid way of distributing the cake we must prove two things: The probabilities of all the combinations are positive and each of these probabilities is less than one.

Each of the probabilities $x_i, ... > 0$ in our formula for the probability of a combination is positive, so the only thing that could make the probability negative is if the x_i not in the combination is greater than the sum of the x_js that are in the combination. If this were the case it would be a contradiction since each x_i can be at most $\frac{1}{10}$ so the remaining probabilities must sum to at least $\frac{9}{10}$.
$\frac{9}{10} - \frac{1}{10} > 0$
This proves the the combinations must always be positive.

The sum of $x_1 + x_2 + ...x_{11}$ is equal to 1 and each $x_i > 0$ which means that no matter which x_i is not included in the combination it, the probability of a size 10 combination will always be less than 1. □

(b) Generalize your solution to distribute k slices to n campers, for all k and n with $n \geq k \geq 1$.

Solution: Claim: We can assign probabilities to each of the possible combinations of ways to give k slices of cake to n kids. We can then create sectors on a roulette wheel for each combination with an area proportional to its probability. We can then spin this roulette wheel to determine which group of kids will receive a slice of cake. Each of the probabilities of combination $C(i,..)$ is given by the formula:

$$\frac{k}{\binom{n-1}{k-1}}((\sum_{j=1}^{n} x_j) - \frac{\binom{n-2}{k-2}}{\binom{n-2}{k-1}}(\sum_{m=1}^{n} x_m)), j \in i_1, ...i_k, m \notin i_1, ..., i_k$$

Proof: To prove this we must prove that each of the probabilities of the combinations are less than 1 and that each probability is positive.
If $\frac{n-1}{n-k} + (\sum_{j=1}^{n} x_j) - (\sum_{m=1}^{n} x_m) < 0$
this is a contradiction to part a so the probability must be positive.
$\frac{n-1}{n-k} + (\sum_{j=1}^{n} x_j) - (\sum_{m=1}^{n} x_m)$ is the largest probability, and was proven to be less than one in part a, so the probability is always less than one.

□

(c) Suppose that, in the setup of part (b), you do not know the value of k (the number of slices). Instead, you will put the Mathcampers in a random order, according to some strategy, and then serve slices of cake in that order until the cake runs out. Is there a way to do this so that for each i, the i^{th} Mathcamper will have a probability of exactly kx_i of getting a slice of cake—no matter what k turns out to be? (You may still assume that we have $0 < x_i < \frac{1}{k}$ for all i, and $x_1 + \cdots + x_n = 1$.)

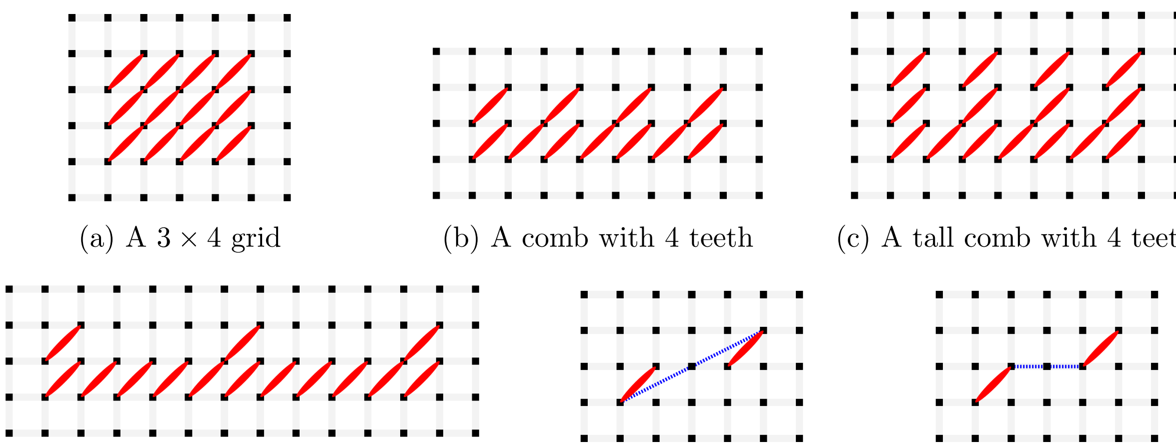
Solution: Observation:We can approach this problem by using the probabilities of the each permutation available instead of each combinations, since the order in which the kids are in line is significant. □

Problem 5

"Half stitching" is a form of embroidery which makes images out of diagonal stitches in a square grid. Our images will all be created from a single thread, which alternates traveling in straight lines (called stitches) along the front and the back of the grid:

- The front of the grid is where the image is formed. Here, each stitch goes from a point (x, y) either to $(x+1, y+1)$ or $(x-1, y-1)$. We say that we *stitch a square* if a stitch on the front follows its diagonal (from top right to bottom left or from bottom left to top right).
- The back of the grid is not part of the image. Here, each stitch can follow any straight line—but it must travel a positive distance, because if you end a stitch where it started, it will unravel.

This problem is about minimizing the total length of thread needed to stitch a given pattern. For example, Figure 9e and Figure 9f (with front stitches in red and back stitches in blue) have the same pattern stitched, but the thread is $2\sqrt{2} + 2\sqrt{5}$ units long in the first case and only $2\sqrt{2} + 2$ units long in the second case.



(a) A 3×4 grid
(b) A comb with 4 teeth
(c) A tall comb with 4 teeth
(d) A wide comb with 3 teeth
(e) Two example stitches
(f) Two example stitches

Figure 9: Diagrams for problem 5

(a) Show that the minimum length of thread (front and back) needed to stitch every square in an $m \times n$ rectangular grid is $mn(\sqrt{2}+1) - 1$. An example with $m = 3$ and $n = 4$ is shown in Figure 9a.

Solution: We can create a graph that is representative of a stitched pattern (Figure 10).
Each vertices/node in the graph represents a front diagonal stitch and each node has two connecting edges to another node. The two connecting edges between nodes in the graph represent the shortest distance from stitch a to stitch b on either end of a and b

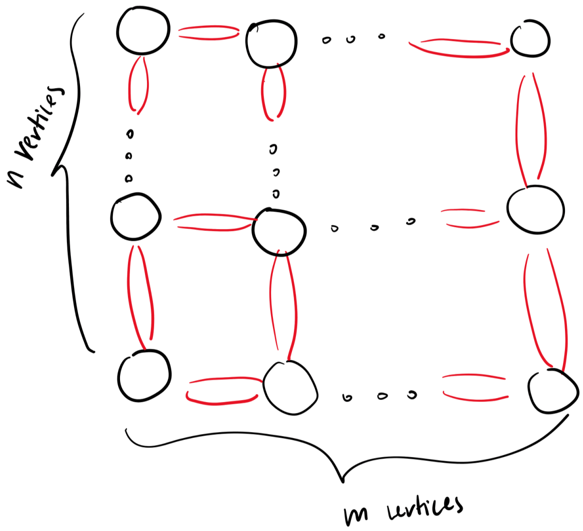


Figure 10: Graph representation of stitched pattern

's stitch. We can notice that each vertices must be visited at least once since we need all the diagonal stitches represented by each node to make the embroidered pattern. Each of the mn vertices also represents $\sqrt{2}$ traveled in order to create the front diagonal stitch.

The restriction of needing alternating front stitches and back stitches is irrelevant with this approach since we can assign each node to be the front stitches, and the connecting edges between nodes to be the back stitches. This means edges will not be allowed to be traversed twice since a stitch will come undone if two back stitches are in the same place. Our problem is now reduced down to: What is the shortest path to traverse all the vertices? The: length of the shortest path + $mn\sqrt{2}$ will be the shortest length of thread needed to create the pattern.

Since we need to visit all the vertices without traversing the same edge twice we must visit at least $mn - 1$ edges since it takes at least $mn - 1$ edges to connect mn vertices.

This can be done for this stitch in particular by starting at the top leftmost vertex, travelling down to the bottom most left vertex, going one right to the next column, going up the column to the top vertex, going right to the next column and repeating the pattern until you have travelled mn vertices (Figure 11).

We can also observe that the minimum length from stitch a's bottom vertex to its adjacent stitch b is equal to the minimum distance between stitch b starting from a's upper vertex. This is because all the diagonals are parallel so as you traverse the stitch,

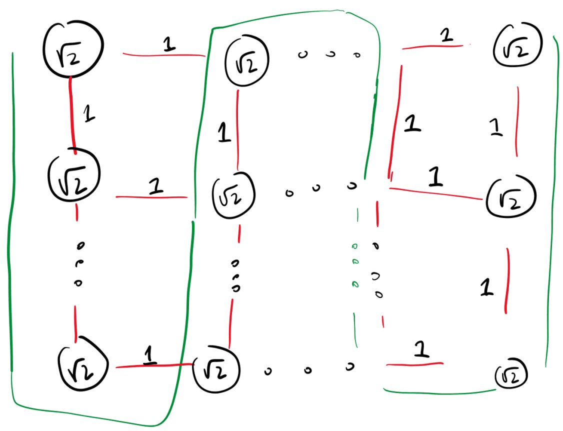


Figure 11: Graph representation with lengths and sample traversal path

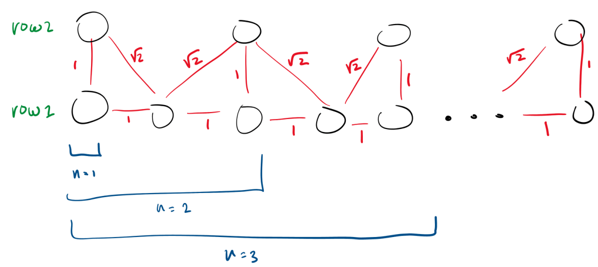


Figure 12: Graph representation for comb pattern

the distance between two stitches stays the same. The graph can now be simplified further as well by reducing the two edges between each vertices to one (Figure 11).
Since each of the diagonals are spaced one unit apart, the minimum distance between each of the vertices is 1. This means all our edges will be of length 1.

Our path length now is $1(mn-1)$ and the length of our front stitches that come from each of the vertices is $mn\sqrt{2}$ so the total length of thread we would need to make the stitch is: $1(mn-1)+mn\sqrt{2} = mn(\sqrt{2}+1)-1$

□

(b) A *comb with n teeth* is a pattern of width $2n-1$ and height 2 that stitches every square in the bottom row and every other square in the top row; an example of this pattern with $n=4$ is shown in Figure 9b. What is the minimum length of thread (front and back) needed to stitch this pattern, in terms of n?

Solution: We can create the graph representation of this stitch as well (Figure 12).
When $n>1$ the graph has a repeating shape with both lengths of 1 and $\sqrt{2}$ as edges. This changes the simplified problem from "What is the minimum path to connect the

vertices?" to "Which path reduces the total amount of $\sqrt{2}$ edges traversed?"

When $n > 1$, the vertices that represent the teeth of the comb in row two of the graph have at most one length 1 edge and either one or two $\sqrt{2}$ edges which means that when traversing to and from these vertices, the minimum edge length you can have is $1 + \sqrt{2}$ unless the vertices is a start or end of the traversal. If a row two vertices is a start or end point of the traversal of the graph then the minimum edge length is 1 since it only needs one edge to either travel to it or away from it. That means in order to minimize the edge length, row 2 vertices must be our start and end points when traversing the graph. Only the starting vertices will need an edge length connecting it since an end vertices does not need to connect to another node via an edge since all vertices have already been traversed. All non-row two vertices will be connected to another vertices with a consecutive length 1 stitch since every non-row two vertices has two length 1 edges connecting to it.

The amount of front stitches or vertices V in a comb pattern of size n is $V = 2+3(n-1)$. The amount of edges needed to connect V vertices is $V - 1$ edges or $1 + 3(n-1)$. The amount of teeth in the pattern or row two vertices in the graph for each n is $n+1$.

The number of $\sqrt{2}$ edges will be $(n+1) - 2 = n - 1$. The number of length 1 edges will be $1 + 3(n-1) - (n-1)$.

That means our minimum length would be:
$1 + 3(n-1) - (n-1) + (n-1)\sqrt{2} = 2n - 1 + \sqrt{2}(n-1)$

□

(c) A *tall comb with n teeth* is a pattern of width $2n - 1$ and height 3 that stitches every square in the bottom row and every other square in the top two rows; an example of this pattern with $n = 4$ is shown in Figure 9c. What is the minimum length of thread (front and back) needed to stitch this pattern, in terms of n?

Solution: We can minimize this problem to the same problem in part b of, how do we minimize the amount of $\sqrt{2}$ edges we travel on. We can notice that on the top right and top left most vertices on any tall comb, the only edges connecting it are of length $\sqrt{2}$ (Figure 13). This means we should set the top most corners as our start and end points to the traversal since that will reduce the use of two $\sqrt{2}$ edges to travel to and from that vertices and instead we would only need one. The end point will not need to have a consecutive edge in the traversal since there is no need to keep traversing the graph after travelling to all the vertices. The row 3 vertices that are not the corners, will require the use of at least one$\sqrt{2}$ edge since it has only one length 1 edge and two $\sqrt{2}$ edges. The rest of the vertices have at least two length 1 edges so the rest of the

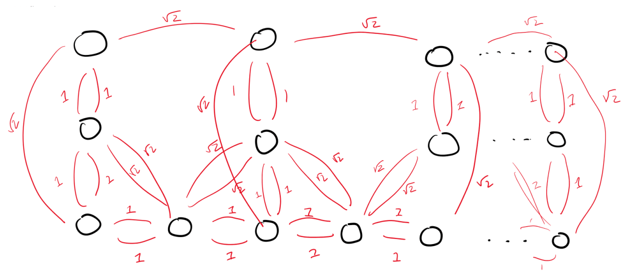


Figure 13: Graph Representation of tall comb

edges will be length 1.
There are $4n-1$ total vertices, so to traverse through all the vertices there needs to be $4n-2$ edges.
There are $n-2$ row 3 vertices that will use at least one $\sqrt{2}$ edge. The rest of the $3n$ edges will have length 1 stitch. Each of the $2n+2n-1$vertices is a $\sqrt{2}$ length front stitch. This means the minimum length of thread is:
$((n-2)+2n+2n-1)\sqrt{2}+3n = 3n+(5n-3)\sqrt{2}$ □

(d) A *wide comb with n teeth* is a pattern of width $5n-4$ and height 2 that stitches every square in the bottom row and every *fifth* square in the top row; an example of this pattern with $n=3$ is shown in Figure 9d. What is the minimum length of thread (front and back) needed to stitch this pattern, in terms of n?

Solution: We can minimize this problem to the same problem in part b: how do we minimize the amount of $\sqrt{2}$ edges we travel on? We can notice that on the row two vertices there are only two edges connecting the vertices to other vertices. One of these lengths is 1 and the other $\sqrt{2}$. In order to minimize the amount of $\sqrt{2}$ edges travelled we can set the end point of the traversal to a corner edge and the start of the traversal to a corner edge. This is because these two vertices will never be forced to connect to another vertices with the length $\sqrt{2}$ edge since they only connect to one other vertices and can use the length 1 edge. The end vertices of the traversal does not need to have a consecutive edge in the traversal as stated in part (c). The rest of the row two vertices are forced to be connected via their $\sqrt{2}$ edge since they only have one length 1 edge and 1 $\sqrt{2}$ edge and they have to connect to two vertices.
There are $6n-4$ vertices so there is at least $6n-5$ edges needed to connect all the vertices.
There are $n-2$ row 2 vertices that use one $\sqrt{2}$ edge, $5n-3$ vertices that use a length 1 edge, and $6n-4$ vertices that have a $\sqrt{2}$ front stitch. This means the minimum length of thread is $(n-2+6n-4)\sqrt{2}+5n-3 = 5n-3+(7n-6)\sqrt{2}$ □

(e) Can you prove anything in general about the minimum length of thread needed to stitch a pattern?

Solution: When the pattern is turned into a graph, the vertices with the largest length edge that is forced to be used, will be assigned as the last vertices in the traversal. That means the stitch that corresponds with that vertices will be the last stitch made when creating that pattern.

Proof: The last node in the traversal of the graph does not to connect to any other nodes so there will be no consecutive edge coming from this vertices. When finding the minimum length of thread, it is minimizing long length edges, and since this edge was going to be the longest and now it is not, this vertices must be the last vertices. $\square$